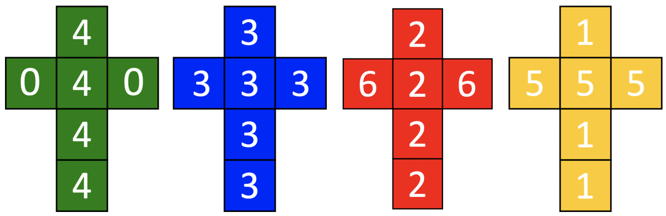
CEMC at Home

Grade 11/12 - Friday, April 3, 2020
A Dicey Situation

In this activity, we will investigate the properties of various non-standard dice.

You Will Need: Two players and four standard six-sided dice.

Description of the Dice: Alter your dice so that the sides of the dice are labelled according to the following diagrams. For example, you could put stickers on each side of the dice.



If you do not have an easy way to alter your dice, you can make use of the following conversion table for each of the dice. This table will allow you to roll a standard die as if it were one of the dice shown above. For example, if you roll a 3 on the standard die representing the green die, then you interpret this as rolling a 4 on the green die (using the second “Green” row in the table).

Die Colour	Number Rolled on Standard Die	Number Rolled on Altered Die
Green	1, 6	0
	2, 3, 4, 5	4
Blue	1, 2, 3, 4, 5, 6	3
Red	1, 6	6
	2, 3, 4, 5	2
Yellow	1, 4, 6	5
	2, 3, 5	1

If all of your dice are identical, you will need a way to keep track of which die is which “colour”.

Investigation:

To start a game, each player chooses a die to use for the entire game. One game consists of 20 rounds. In each round, each player rolls their die and the player that rolls the higher number wins the round. The winner of the game is the person who won the most rounds. No round can end in a tie, but the game may end in a tie. Play the game a number of times and alternate the pair of dice used for each game. Keep track of the scores of each game, along with which player had which die for each game.

Follow-up Questions:

1. If you choose your die first, is there a best choice for your die?
2. If you choose your die second, is there a best choice for your die?
   Your answers can depend on what die the other player chooses.
3. Try to justify your answers for 1. and 2. by analyzing the four dice and calculating probabilities.

More Info:

Check out the CEMC at Home webpage on Tuesday, April 14 for the solution to A Dicey Situation.

To refresh your knowledge of probabilities, check out this lesson from the CEMC courseware.

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A Dicey Situation - Solution

Investigation:

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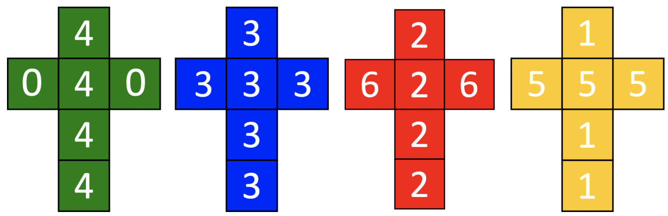

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Follow-up Questions:

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3. Try to justify your answers for 1. and 2. by analyzing the four dice and calculating probabilities.

Solution: You may have answered 1. and 2. using information gathered during your investigation. We will give answers to 1. and 2. that are based upon theoretical probabilities related to the four dice. We start by examining the game that arises if certain pairs of dice are chosen by the two players. Note that the rolls of the two dice are independent.

Green Versus Blue

What happens if the two players pick the green and blue dice (in some order)? Since each of the dice has 6 faces, and each face is equally likely to end up as the top face on a roll, there are $6 \times 6 = 36$ equally likely outcomes when these two dice are rolled. We indicate which die wins in each of the 36 cases the table below:

	3	3	3	3	3	3
0	B	B	B	B	B	B
0	B	B	B	B	B	B
4	G	G	G	G	G	G
4	G	G	G	G	G	G
4	G	G	G	G	G	G
4	G	G	G	G	G	G

Based on this table, we can see that the probability of green winning is

$$\text{P(Green Winning Against Blue)} = \frac{24}{36} = \frac{2}{3}$$

and the probability of blue winning is

$$\text{P(Blue Winning Against Green)} = \frac{12}{36} = \frac{1}{3}$$

Since rolls of the two dice are independent, we can calculate probabilities without listing the possible outcomes. We know that the blue die will always roll a 3, and so we only need to consider the roll of the green die. The probability that the green die rolls a 0 is $\frac{2}{6} = \frac{1}{3}$, and the probability that the green die rolls a 4 is $\frac{4}{6} = \frac{2}{3}$. If the green die rolls a 0, then the blue die wins, and if the green die rolls a 4, then the green die wins. Therefore, we have

$$\text{P(Green Winning Against Blue)} = \frac{2}{3}$$

$$\text{P(Blue Winning Against Green)} = \frac{1}{3}$$

So between green and blue, green is a better choice.

Blue Versus Red

What happens if the two players pick the blue and red dice (in some order)? We know that the blue die will always roll a 3, and so we only need to consider the roll of the red die. If the red die rolls a 2, which happens with probability $\frac{4}{6} = \frac{2}{3}$, then the blue die wins. If the red die rolls a 6, which happens with probability $\frac{2}{6} = \frac{1}{3}$, then the red die wins. Therefore, we have

$$\text{P(Blue Winning Against Red)} = \frac{2}{3}$$

$$\text{P(Red Winning Against Blue)} = \frac{1}{3}$$

So between blue and red, blue is the better choice.

Red Versus Yellow

What happens if the two players pick the red and yellow dice (in some order)? In the table below we indicate which die wins for each of the 36 equally likely rolls.

	1	1	1	5	5	5
2	R	R	R	Y	Y	Y
2	R	R	R	Y	Y	Y
2	R	R	R	Y	Y	Y
2	R	R	R	Y	Y	Y
6	R	R	R	R	R	R
6	R	R	R	R	R	R

From this table, we see that

$$\text{P(Red Winning Against Yellow)} = \frac{24}{36} = \frac{2}{3}$$

$$\text{P(Yellow Winning Against Red)} = \frac{12}{36} = \frac{1}{3}$$

Since the rolls of the two dice are independent, we can calculate probabilities without listing all of the outcomes. There is only one type of roll that will result in a win for the yellow die: 5 rolled on the yellow die and 2 rolled on the red die. The probability that a 5 is rolled on the yellow die is $\frac{3}{6} = \frac{1}{2}$, and the probability that a 2 is rolled on the red die is $\frac{4}{6} = \frac{2}{3}$.
Multiplying these two probabilities we determine that

$$\text{P(Yellow Winning Against Red)} = \left(\frac{1}{2}\right)\left(\frac{2}{3}\right) = \frac{1}{3}$$

and since there are only two possibilities (yellow wins or red wins), the two probabilities must sum to 1, and so we have that

$$\text{P(Red Winning Against Yellow)} = 1 - \frac{1}{3} = \frac{2}{3}.$$

So between red and yellow, red is the better choice.

Yellow Versus Green

What happens if the two players pick the yellow and green dice (in some order)? Similar reasoning to the case above allows us to determine that

$$\text{P(Green Winning Against Yellow)} = \left(\frac{2}{3}\right)\left(\frac{1}{2}\right) = \frac{1}{3}$$
$$\text{P(Yellow Winning Against Green)} = 1 - \frac{1}{3} = \frac{2}{3}$$

So between yellow and green, yellow is the better choice.

At this point there are two more pairs to consider (green and red, and blue and yellow) which are left to you. Let's summarize our findings for the four pairs we considered:

- In a game played with the green and blue dice, the better die is green.
- In a game played with the blue and red dice, the better die is blue.
- In a game played with the red and yellow dice, the better die is red.
- In a game played with the yellow and green dice, the better die is yellow.

One interesting thing that our results show is that the dice form a kind of "cycle", rather than an ordering from "best" to "worst". For every choice of die, there exists another die which is a better choice, and another die that is a worse choice. Now we discuss the follow-up questions:

If you choose your die first, then no matter what die you choose, the other player will be able to choose a die that beats your die with probability $\frac{2}{3}$. If you choose your die second, then the best choice of die is outlined above: if the first player chooses blue, then you should choose green; if the first player chooses red, then you should choose blue; if the first player chooses yellow then you should choose red, and if the first player chooses green then you should choose yellow. Can you convince yourself that these are the best choices? For example, if your opponent chooses the red die, then we know the blue die is a good choice (from bullet 2) and the yellow die is a bad choice (from bullet 3), but what about the green die? Since the blue die will win against the red die with a probability of $\frac{2}{3}$, you need to convince yourself that the green die has worse odds than this against the red die.

Background of the dice: These dice were invented by Bradley Efron, an American statistician. The dice provide an example of four *non-transitive** dice. You can create different sets of non-transitive dice. For example, you can create a set of three 6-sided non-transitive dice using the values from 1 to 9 on the faces of the dice. Non-transitive dice are an interesting topic about which you might want to do more reading.

*You may have heard the word *transitive* used in math before, and even if you have not, you have likely worked with many comparisons that are transitive: for example, if $x < y$ and $y < z$ then we must have $x < z$. In the case of Efron's dice, we can compare the dice, but these comparisons may not be transitive: for example, if "green is better than blue" and "yellow is better than green", must "yellow be better than blue"?