CEMC at Home

Grade 11/12 - Tuesday, March 24, 2020
Divisors and Primes

There are lots of problems that involve divisors of integers: counting divisors, looking for particular divisors, identifying common divisors, and more. For the following problems it might be helpful to review what a prime number is and how to find the prime factorization of an integer. Let's practice:

1. Find the prime factorization of 72 600.

   To help towards a solution, think about the following questions:

   - What are prime numbers?
   - Is 2 a divisor of 72 600? Is 3 a divisor of 72 600?
   - For each prime divisor p of 72 600, how many copies of p can we factor out of 72 600?

2. For how many integers n is $72\left(\frac{3}{2}\right)^n$ equal to an integer?

   To help towards a solution, think about the following questions:

   - Try some values of n. What if $n = 1$? What if $n = 10$? What if $n = -4$?
   - How big can n be? How small can n be?
   - Could prime factorizations help us here?

3. Determine the number of positive divisors of the integer 14!.

   Note: The *factorial* of a positive integer n, denoted by $n!$, is the product of all positive integers less than or equal to n. For example, $4! = 1 \times 2 \times 3 \times 4 = 24$.

4. For a positive integer n, $f(n)$ is defined as the largest power of 3 that is a divisor of n.

   What is $f\left(\dfrac{100!}{50!20!}\right)$?

More Info:

Check the CEMC at Home webpage on Wednesday, March 25 for a solution to Divisors and Primes. We encourage you to discuss your ideas online using any forum you are comfortable with.

These problems were taken from the CEMC's free online course *Problem Solving and Mathematical Discovery*. Check it out here: `https://courseware.cemc.uwaterloo.ca/40`

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Divisors and Primes - Solution

1. Find the prime factorization of 72 600.

   Solution:

   First, we factor 72 600 into two factors, $72\,600 = 726 \times 100$. Next, we factor each of these factors into a product of two factors, $726 = 2 \times 363$ and $100 = 10 \times 10$. We repeat this process until all the factors are prime numbers, (some of these prime factors will be repeated),

$$\begin{aligned} 72\,600 &= 726 \times 100 \\ &= 2 \times 363 \times 10 \times 10 \\ &= 2 \times 3 \times 121 \times 2^2 \times 5^2 \\ &= 2 \times 3 \times 11^2 \times 2^2 \times 5^2 \\ &= 2^3 \times 3 \times 5^2 \times 11^2. \end{aligned}$$

2. For how many integers n is $72\left(\frac{3}{2}\right)^n$ equal to an integer?

   Solution:

   Notice that the prime factorization of 72 is $2^3 \times 3^2$, so the expression $72\left(\frac{3}{2}\right)^n$ can be written as

$$72\left(\frac{3}{2}\right)^n = 2^{3-n} \times 3^{2+n}.$$

   The expression will be an integer whenever the exponents $3 - n$ and $2 + n$ are non-negative integers. So, $3 - n \geq 0$ and $2 + n \geq 0$ imply that $n \leq 3$ and $n \geq -2$. Hence, there are six possible values of n, which are $-2, -1, 0, 1, 2, 3$.

3. Determine the number of positive divisors of the integer 14!.

   Solution:

   First, we find the prime factorization of 14!, which is the following:

$$14! = 2^{11} \times 3^5 \times 5^2 \times 7^2 \times 11 \times 13.$$

   Any divisor of 14! has a prime factorization of the form $2^p \times 3^q \times 5^r \times 7^s \times 11^t \times 13^u$, where p, q, r, s, t, u are integers and $0 \leq p \leq 11$, $0 \leq q \leq 5$, $0 \leq r \leq 2$, $0 \leq s \leq 2$, $0 \leq t \leq 1$, and $0 \leq u \leq 1$.