# CEMC at Home

## Grade 11/12 - Tuesday, March 31, 2020
## Making a List

Ellie has two lists, each consisting of 6 consecutive positive integers. The smallest integer in the first list is $a$, the smallest integer in the second list is $b$, and $a < b$. Ellie makes a third list which consists of the 36 integers formed by multiplying each number from the first list with each number from the second list. (This third list may include some repeated numbers.)

1. Suppose that Ellie starts with $a = 5$ and $b = 16$. Can you determine the four numbers that each appear twice in the third list without writing out all 36 numbers in the third list?

2. Suppose that Ellie's third list has the following properties:

   (i) the integer 49 appears in the third list,

   (ii) there is no number in the third list that is a multiple of 64, and

   (iii) there is at least one number in the third list that is larger than 75.

   Determine all possibilities for the pair $(a, b)$.

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**More Info:**

Check out the CEMC at Home webpage on Tuesday, April 7 for the solution to Making a List.

Part of this question appeared on a past Euclid Contest. You can see the original question and the rest of the contest here: 2015 Euclid Contest.

# CEMC at Home

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## Making a List - Solution

**Problem:**

Ellie has two lists, each consisting of 6 consecutive positive integers. The smallest integer in the first list is $a$, the smallest integer in the second list is $b$, and $a < b$. Ellie makes a third list which consists of the 36 integers formed by multiplying each number from the first list with each number from the second list. (This third list may include some repeated numbers.)

1. Suppose that Ellie starts with $a = 5$ and $b = 16$. Can you determine the four numbers that each appear twice in the third list without writing out all 36 numbers in the third list?

2. Suppose that Ellie's third list has the following properties:

   (i) the integer 49 appears in the third list,

   (ii) there is no number in the third list that is a multiple of 64, and

   (iii) there is at least one number in the third list that is larger than 75.

   Determine all possibilities for the pair $(a, b)$.

**Solution to 1.**

The first list is 5, 6, 7, 8, 9, 10 and the second list is 16, 17, 18, 19, 20, 21.

The prime numbers that are a factor of at least one number in the first two lists are as follows:

$$2, 3, 5, 7, 17, 19$$

We examine the 36 products in the third list based on their prime factors.

*Products with a factor of 17 or 19*

Note that 17 is the only number in the first or second list that has a factor of 17. This means that there will be exactly six numbers in the third list that have a factor of 17 (obtained by multiplying each number in the first list by 17) and these six numbers will be all be different. The situation is similar for the six numbers in the third list that have a factor of 19. We can see that all twelve of these numbers only appear once in the third list.

*Products with a factor of 7*

Note that 7 and 21 are the only numbers in the first or second list that have a factor of 7. Suppose we have $7 \times m = n \times 21$ where $m \neq 21$ comes from the second list and $n \neq 7$ comes from the first list. Since $m$ must have a factor of 3, the only possibility for $m$ is 18, which forces $n$ to be 6, and gives

$$7 \times 18 = 6 \times 21 = 126$$

*Products with a factor of 5*

Note that 5, 10, and 20 are the only numbers in the first or second list that have a factor of 5. Suppose we have $5 \times m = n \times 20$ where $m \neq 20$ comes from the second list and $n \neq 5$ comes from the first list. Since $m$ must have a factor of 4, the only possibility for $m$ is 16, which would force $n$ to be 4, which is not possible as 4 is not in the first list. Now suppose we have $10 \times m = n \times 20$ where $m \neq 20$ comes from the second list and $n \neq 10$ comes from the first list. Since $m$ must have a factor of 2, the only possibilities for $m$ are 16 and 18, which would force $n$ to be 8 and 9, respectively, giving us

$$10 \times 16 = 8 \times 20 = 160$$

$$10 \times 18 = 9 \times 20 = 180$$

*Products whose only prime factors are 2 or 3*

All remaining numbers in the list must be formed by multiplying one of 6, 8, or 9 by one of 16, or 18. There is only one final duplication here, formed by

$$8 \times 18 = 9 \times 16 = 144$$

Therefore the four numbers that each appear twice in the third list are as follows:

$$\begin{aligned} 126 &= 7 \times 18 = 6 \times 21 \\ 144 &= 8 \times 18 = 9 \times 16 \\ 160 &= 8 \times 20 = 10 \times 16 \\ 180 &= 9 \times 20 = 10 \times 18 \end{aligned}$$

**Solution to 2.**

We will start by considering what condition (i) tells us about the values of $a$ and $b$ as it seems to be the most restrictive of the three conditions.

Condition (i) tells us that 49 must be a product of an integer from the first list and an integer from the second list. Since $49 = 7^2$, 7 is prime, and all integers in the two lists are positive, these integers must be either 1 and 49 or 7 and 7. We will find all possible values of $a$ and $b$ by considering two cases separately:

- Case 1: 49 was obtained in the third list by multiplying 1 and 49
- Case 2: 49 was obtained in the third list by multiplying 7 and 7

*Note: It is not possible for 49 to be obtained in both of these ways at once because if a list contains 49 then it cannot also contain 7. However, knowing this will not be important for our solution.*

Case 1: 49 was obtained by multiplying 1 and 49.

Since the number 1 is in one of the lists, we must have either $a = 1$ or $b = 1$. The condition of $a < b$ means we must have $a = 1$. This means that the first list must be

$$1, 2, 3, 4, 5, 6$$

and the number 49 must appear somewhere in the second list.

Therefore, the second list is one of the following six lists (with each list appearing horizontally):

$$\begin{array}{cccccc} 44, & 45, & 46, & 47, & 48, & 49 \\ 45, & 46, & 47, & 48, & 49, & 50 \\ 46, & 47, & 48, & 49, & 50, & 51 \\ 47, & 48, & 49, & 50, & 51, & 52 \\ 48, & 49, & 50, & 51, & 52, & 53 \\ 49, & 50, & 51, & 52, & 53, & 54 \end{array}$$

Notice that $4 \times 48 = 192 = 64 \times 3$. Since 4 is in the first list, and no number in the third list can be a multiple of 64, the second list cannot contain the number 48. This leaves just one possibility for the second list (the last one above):

$$49, 50, 51, 52, 53, 54$$

This case leads us to exactly one possibility for the pair $(a, b)$, namely $(1, 49)$.

We can verify that the third list for the pair $(a, b) = (1, 49)$ actually satisfies conditions (ii) and (iii). For (ii), we note that $64 = 2^6$ and that we can get at most two factors of 2 from a number in the first list and at most two factors of 2 from a number in the second list. It follows that any product in the third list will have at most 4 factors of 2, and hence cannot be a multiple of 64. For (iii), we note that $2 \times 49 = 98$ is in the third list and is greater than 75.

Case 2: 49 was obtained by multiplying 7 and 7.

In this case, we know that the number 7 must appear in both the first list and the second list. In order for this to happen we need to have $2 \leq a \leq 7$ and $2 \leq b \leq 7$. Since $a < b$, we actually must have $3 \leq b \leq 7$. (The smallest $a$ can be is 2 and so $b$ must be at least one more than that.)

Since $3 \leq b \leq 7$, the second list *must* contain the number 8. This means that to satisfy condition (ii), the first list *cannot* contain the number 8. Therefore, we must have $a = 2$, making the first list

$$2, 3, 4, 5, 6, 7$$

Since $7 \times 10 = 70$ and $7 \times 11 = 77$, the third list can only satisfy condition (iii) if the second list contains a number at least as large as 11. This means we cannot have $b = 3$, $b = 4$, or $b = 5$, leaving the only possible values to be $b = 6$ or $b = 7$. These values produce the following second lists:

$$b = 6: \quad 6, 7, 8, 9, 10, 11$$

$$b = 7: \quad 7, 8, 9, 10, 11, 12$$

Therefore, this case leads us to two additional possibilities for the pair $(a, b)$, namely $(2, 6)$ and $(2, 7)$.

We can verify that the third list for each of the the pairs $(a, b) = (2, 6)$ and $(a, b) = (2, 7)$ satisfies conditions (ii) and (iii) using a similar argument to the one given in Case 1.

Combining the two cases, we conclude that there are exactly three pairs, $(a, b)$, that satisfy all three conditions. They are as follows:

$$(1, 49), (2, 6), (2, 7)$$