CEMC at Home

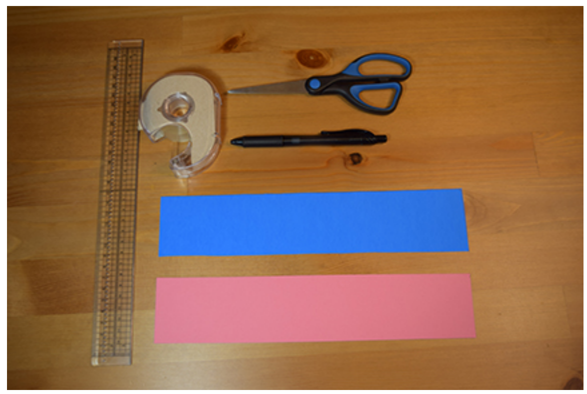
Grade 11/12 - Friday, March 27, 2020
The Möbius Strip

A Möbius strip is a mathematical object that has interesting properties. It is a surface with only one face (or side) and only one edge (or boundary). In this activity we will build a Möbius strip and investigate its curious properties. The purpose of this activity is not to fully understand the mathematics of a Möbius strip, but rather to hopefully surprise and intrigue you!

You will need:

- A pencil
- A ruler
- Scissors
- Tape
- Two rectangular strips of paper of different colours.

  Strips of around 6 cm wide and 30 cm long will work well. We will use one blue strip and one pink strip, but you can use any colours you want.



How to construct a Möbius strip:

1. Use your ruler to draw a line along the blue strip that divides the strip into two equal parts. Do the same on the other side of the strip.
2. Use your ruler to draw two lines along the pink strip that divides the strip into three equal parts. Do the same on the other side of the strip.
3. Grab the blue strip by the two short edges. Twist one end of the strip half of the way around and join the two short edges together. (Make sure this is a "half twist" and not a "full twist".) Line up the short edges and tape them together from end to end.
4. Repeat the same process with the pink strip. You now have two Möbius strips.

Let's explore some properties of our Möbius strips!

1. Take one of the strips you made and answer the following questions:

   (a) How many faces does the Möbius strip have?

   You might need to spend some time thinking about what is meant by a "face" here.

   (b) How many edges does the Möbius strip have?

   You might need to spend some time thinking about what is meant by an "edge" here.

   (c) Does the Möbius strip have an "inside" and an "outside"?

2. Take the blue Möbius strip and answer the following questions:

   (a) What do you think will happen if you cut the strip along the line drawn in the middle of the strip? How many detached pieces do you think you will you get? Will they be Möbius strips? Make your predictions.

   (b) Let's verify your predictions. Cut the blue strip along the middle line. You will need to carefully cut or puncture the strip somewhere along this line in order to start the cut. What happens once you cut along this line? Is it what you predicted? How many edges does each detached piece have? How many faces?

   (c) After making your cut, you may have ended up with an object that surprised you. Looking back, can you explain why you ended up with this object?

3. Take the pink Möbius strip and answer the following questions:

   (a) What do you think will happen if you cut the strip along one of the two lines that we drew down down the strip? How many detached pieces do you think you will you get? Will they be Möbius strips? Make your predictions.

   (b) Let's verify your predictions. Cut the pink strip along one of the lines. When you cut, you might notice that it doesn't actually matter which of the two lines you chose to cut along. Is the result what you predicted? How many edges does each detached piece have? How many faces?

   (c) After making your cut, you may have ended up with an object that surprised you. Looking back, can you explain why you ended up with this object?

More Info:

Check the CEMC at Home webpage on Friday, April 3 for further discussion on The Möbius Strip.

CEMC at Home

Grade 11/12 - Friday, March 27, 2020
The Möbius Strip - Solution

1. Take one of the strips you made and answer the following questions:

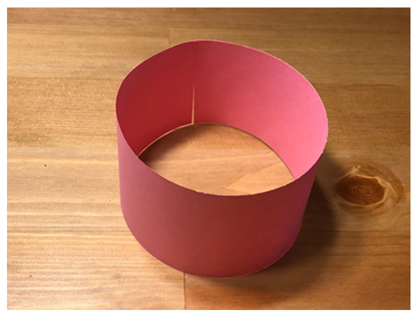
   (a) How many faces does the Möbius strip have?

   (b) How many edges does the Möbius strip have?

   (c) Does the Möbius strip have an "inside" and an "outside"?

   Discussion:

   We will compare our Möbius strip with a cylinder. A cylinder can be made by taking the two short edges of a strip, lining them up, and taping them together from end to end.

   A cylinder has an "outside" face and an "inside" face. An ant walking on one of these faces must cross an edge (or boundary) to get to the other face. The Möbius strip has only one face (or side). An ant can walk along the entire surface of the Möbius strip without crossing an edge (or boundary). In particular, an ant that begins on any part of either line we drew, can follow this line and will end up back where it started. In doing this, it will have travelled the full length of *both* lines drawn on the two sides of the original blue strip.

   Similarly, a cylinder has a "top" edge and "bottom" edge, but a Möbius strip has only one edge (or boundary). Imagine the ant walking along the edge of the Möbius strip. The ant will travel along the entire edge of the strip and will end up back where it started.

   In summary, a cylinder has two faces and two edges, but a Möbius strip only has one face and one edge. Both can be created from a single strip of paper.

2. Take the blue Möbius strip and answer the following questions:

   (a) What do you think will happen if you cut the strip along the line drawn in the middle of the strip? How many detached pieces do you think you will you get? Will they be Möbius strips? Make your predictions.

   (b) Let's verify your predictions. Cut the blue strip along the middle line. You will need to carefully cut or puncture the strip somewhere along this line in order to start the cut. What happens once you cut along this line? Is it what you predicted? How many edges does each detached piece have? How many faces?

   (c) After making your cut, you may have ended up with an object that surprised you. Looking back, can you explain why you ended up with this object?

Discussion:

Following our intuition with the cylinder, we know that if we cut the cylinder along a middle line parallel to its two edges, then we will obtain two smaller detached cylinders. However, if we cut the Möbius strip along the middle line, the result might surprise us in two ways:

- We get one strip instead of two detached pieces.
- The strip we get is not a Möbius strip!

This result is illustrated in Figure 1.

Figure 1

One way to help us understand this is to think about the process of "gluing and cutting" in a different order. We can *begin* by cutting along the middle line and taping the two pieces together leaving a gap to show our cut. *After* doing this, we can do a "half twist" and join the short edges together as in the original instructions. This is illustrated in Figure 2. Notice that the green ends connect to each other and the black ends also connect to each other.

Figure 2

Try to use the analogy of an ant walking on the surface and edges of the resulting strip to convince yourself that this strip is an attached single piece with two faces and two sides. An effective way to do this is to do the construction shown in Figure 2 yourself and follow the path an ant might take with your finger. Interestingly, this resulting strip is what we would get if we followed the original instructions using two "full twists" instead of a "half twist".

3. Take the pink Möbius strip and answer the following questions:

   (a) What do you think will happen if you cut the strip along one of the two lines that we drew down down the strip? How many detached pieces do you think you will you get? Will they be Möbius strips? Make your predictions.

   (b) Let's verify your predictions. Cut the pink strip along one of the lines. When you cut, you might notice that it doesn't actually matter which of the two lines you chose to cut along. Is the result what you predicted? How many edges does each detached piece have? How many faces?

   (c) After making your cut, you may have ended up with an object that surprised you. Looking back, can you explain why you ended up with this object?

Discussion:

As before, Figure 3 is an illustration of the result and Figure 4 illustrates what happens if we *begin* by cutting and taping before doing a "half twist" and joining the short edges together.

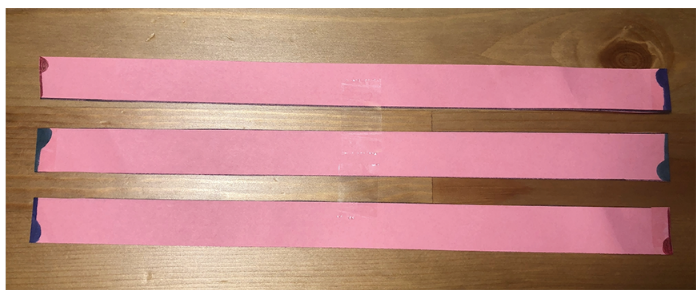


Figure 3

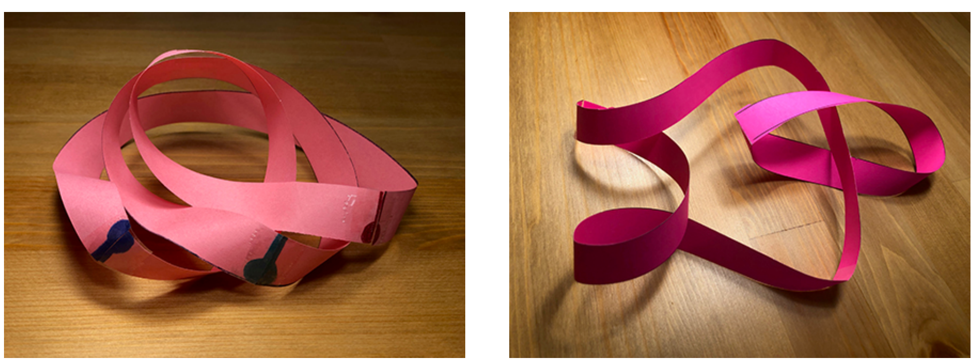


Figure 4

Some interesting and surprising things happen:

- The opposite ends of the upper and lower strips connect to each other making a longer strip which is similar in structure (but narrower) than the one we obtained from the blue strip.
- We also get a shorter strip which is a Möbius strip.
- These two strips are linked together!