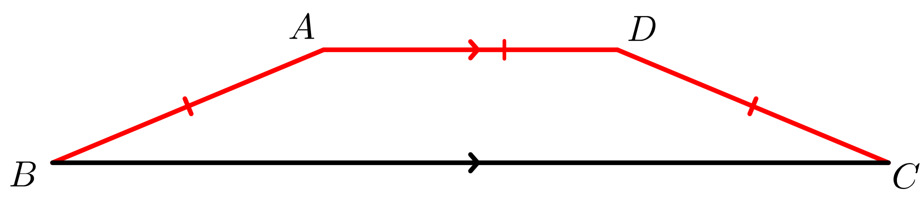
CEMC at Home features Problem of the Week

Grade 11/12 - Thursday, April 2, 2020

Three Equal Sides

In trapezoid $ABCD$, the lengths of AB, AD and DC are equal and the length of BC is 2 units less than the sum of the lengths of the other three sides.



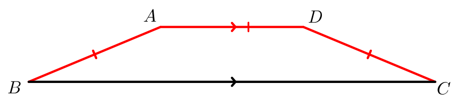
If the distance between the parallel sides AD and BC is 5 units, what is the area of the trapezoid?

More Info:

Check the CEMC at Home webpage on Thursday, April 9 for the solution to this problem. Alternatively, subscribe to Problem of the Week at the link below and have the solution, along with a new problem, emailed to you on Thursday, April 9.

This CEMC at Home resource is the current grade 11/12 problem from Problem of the Week (POTW). POTW is a free, weekly resource that the CEMC provides for teachers, parents, and students. Each week, problems from various areas of mathematics are posted on our website and e-mailed to our subscribers. Solutions to the problems are e-mailed one week later, along with a new problem. POTW is available in 5 levels: A (grade 3/4), B (grade 5/6), C (grade 7/8), D (grade 9/10), and E (grade 11/12).

To subscribe to Problem of the Week and to find many more past problems and their solutions visit: https://www.cemc.uwaterloo.ca/resources/potw.php



Problem of the Week
Problem E and Solution
Three Equal Sides

Problem

In trapezoid $ABCD$, the lengths of AB, AD and DC are equal and the length of BC is 2 units less than the sum of the lengths of the other three sides. If the distance between the parallel sides AD and BC is 5 units, what is the area of the trapezoid?

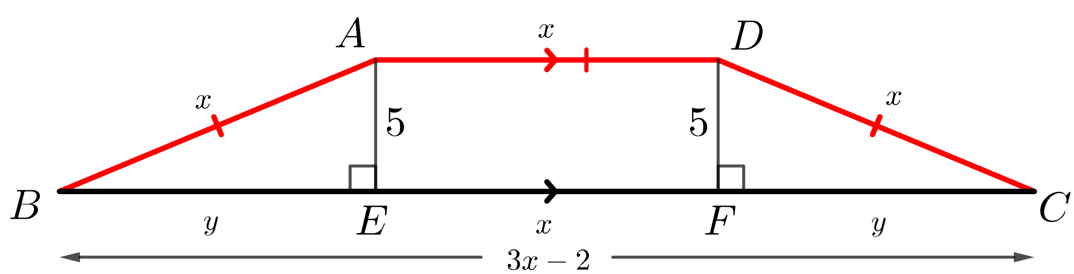
Solution

Let x represent the length of AB. Then $AB = AD = DC = x$. Since the base BC is two less than the sum of the three equal sides, $BC = 3x - 2$.

Construct altitudes from A and D meeting BC at E and F, respectively. Then $AE = DF = 5$, the distance between the two parallel sides.

Let y represent the length of BE. We can show that $BE = FC$ using the Pythagorean Theorem as follows: $BE^2 = AB^2 - AE^2 = x^2 - 5^2 = x^2 - 25$ and $FC^2 = DC^2 - DF^2 = x^2 - 5^2 = x^2 - 25$. Then $FC^2 = x^2 - 25 = BE^2$, so $FC = BE = y$ since $FC > 0$.

Since $\angle AEF = \angle DFE = 90^\circ$ and AD is parallel to EF, it follows that $\angle DAE = \angle ADF = 90^\circ$ and $AEFD$ is a rectangle so $EF = AD = x$. The following diagram contains all of the given and found information.



We can now determine a relationship between x and y.

$$\begin{aligned} BC &= BE + EF + FC \\ 3x - 2 &= y + x + y \\ 2x - 2 &= 2y \\ x - 1 &= y \end{aligned}$$

In right $\triangle ABE$, $AB^2 = BE^2 + AE^2$ gives $x^2 = y^2 + 5^2$. Substituting $y = x - 1$ we get $x^2 = (x-1)^2 + 25$. Solving, $x^2 = x^2 - 2x + 1 + 25$, or $2x = 26$, or $x = 13$.

Since $x = 13$, $3x - 2 = 3(13) - 2 = 37$. Therefore, $AD = x = 13$ and $BC = 3x - 2 = 37$.

Therefore,

$$\begin{aligned} \text{area of trapezoid } ABCD &= AE \times (AD + BC) \div 2 \\ &= 5 \times (13 + 37) \div 2 \\ &= 125 \text{ units}^2 \end{aligned}$$